

# Shifted inverse determinant sums and new bounds for the DMT of space-time lattice codes

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**Abstract**—This paper considers shifted inverse determinant sums arising from the union bound of the pairwise error probability for space-time codes in multiple-antenna fading channels. Previous work by Vehkalahti et al. focused on the approximation of these sums for low multiplexing gains, providing a complete classification of the inverse determinant sums as a function of constellation size for the most well-known algebraic space-time codes. This work aims at building a general framework for the study of the shifted sums for all multiplexing gains. New bounds obtained using dyadic summing techniques suggest that the behavior of the shifted sums does characterize many properties of a lattice code such as the diversity-multiplexing gain trade-off, both under maximum-likelihood decoding and infinite lattice naive decoding. Moreover, these bounds allow to characterize the signal-to-noise ratio thresholds corresponding to different diversity gains.

## I. INTRODUCTION

Shifted inverse determinant sums appear naturally when analyzing the union bound for the pairwise error probability (PEP) of space-time codes over MIMO channels [1]. The high-SNR approximation of these sums was analyzed in [2], providing general bounds on the performance of algebraic space-time codes from division algebras and number fields. In particular, it was shown that the approximate sums are enough to characterize the diversity-multiplexing gain trade-off (DMT) [3] of these codes in the multiplexing gain range  $r \in [0, 1]$ . However, in order to study the DMT for higher multiplexing gains  $r$ , it becomes necessary to consider the original shifted determinant sums. In this work we provide a general framework to analyze shifted sums, which are able to predict the correct DMT curve for  $r > 1$  in some cases. We also discuss the characterization of the “high SNR” threshold as a function of constellation size.

Moreover, we show that while their high-SNR approximations never converge, shifted sums always converge if the number of receive antennas is large enough; this provides new bounds on the DMT performance of naive lattice decoding.

Inverse determinant sums in the sense we are discussing were considered by Tavildar and Viswanath in [4], where the authors analyzed the DMT of several simple space-time codes. The most recent appearance of these sums is in the work of Belfiore and Oggier concerning the eavesdropper’s error probability in the MIMO wiretap channel [5].

Our take on the subject follows the general setting of [2], but we replace the approximation of PEP by the more accurate version. The idea to consider symmetric polynomials and their relations to analyze PEP was given in [6].

### A. Matrix Lattices and spherically shaped coding schemes

Before we can introduce inverse determinant sums, we need a few definitions.

*Definition 1.1:* A matrix lattice  $L \subseteq M_{n \times T}(\mathbb{C})$  has the form

$$L = \mathbb{Z}B_1 \oplus \mathbb{Z}B_2 \oplus \cdots \oplus \mathbb{Z}B_k,$$

where the matrices  $B_1, \dots, B_k$  are linearly independent over  $\mathbb{R}$ , i.e., form a lattice basis, and  $k$  is called the *rank* or the *dimension* of the lattice.

*Definition 1.2:* If the minimum determinant of the lattice  $L \subseteq M_{n \times T}(\mathbb{C})$  is non-zero, i.e.  $\inf_{\mathbf{0} \neq \mathbf{X} \in L} |\det(\mathbf{X}\mathbf{X}^*)| > 0$ , we say that the lattice satisfies the *non-vanishing determinant* (NVD) property.

We now consider a spherical shaping scheme based on a  $k$ -dimensional lattice  $L$  in  $M_{n \times T}(\mathbb{C})$ . Given  $M > 0$  we define

$$L(M) = \{\mathbf{a} \in L : \|\mathbf{a}\|_F \leq M, \mathbf{a} \neq \mathbf{0}\}.$$

Here  $\|\cdot\|_F$  refers to the Frobenius norm.

### B. Motivation and problem statement

Let us suppose that we are considering the complex Gaussian channel and a finite code  $L(M) \in \mathbb{C}^n$ . If the codewords are sent equiprobably, we can upper bound the average error probability by the sum

$$P_e \leq \sum_{\mathbf{x} \in L, 0 < \|\mathbf{x}\|_E \leq 2M} e^{-\|\mathbf{x}\|^2},$$

where the term  $2M$  follows from the fact that we have to consider differences of codewords. The right-hand-side is then a well known truncated *exponential sum* taking values on lattice points of  $L$ . Let us now describe the analogous bound in the fading channel.

Suppose that we have a lattice  $L \subset M_{n \times T}(\mathbb{C})$  and that we have chosen a finite code  $L(M)$  and a constant  $\theta$  such that  $\theta L(M)$  has average energy 1.

Consider the Rayleigh block fading MIMO channel with

$n = n_t$  transmit and  $n_r$  receive antennas. The channel is assumed to be fixed for a block of  $T$  channel uses, but to vary in an independent and identically distributed (i.i.d.) fashion from one block to another. Thus, the channel input-output relation can be written as

$$Y = \sqrt{\frac{\rho}{n}} H \theta X + N, \quad (1)$$

where  $H \in M_{n_r \times n}(\mathbb{C})$  is the channel matrix and  $N \in M_{n_r \times T}(\mathbb{C})$  is the noise matrix. The entries of  $H$  and  $N$  are assumed to be i.i.d. zero-mean complex circular symmetric Gaussian random variables with variance 1. The matrix  $X \in L(M)$  is the transmitted codeword, and the term  $\rho$  denotes the signal-to-noise ratio (SNR).

Following [1], we can upper bound the pairwise error probability between two codewords  $X \neq X'$ , when transmitting with SNR  $\rho$ , as follows:

$$P(\rho, X \rightarrow X') \leq \frac{1}{(\det(I + \frac{\rho \theta^2}{4n} (X - X')(X - X')^*))^{n_r}},$$

where  $*$  denotes complex conjugate transpose. The scaling factor  $4n$  for the SNR  $\rho$  is irrelevant for our asymptotic analysis so we will omit it in the sequel.

We can upperbound the average error probability, when transmitting a codeword from  $L(M)$ , as

$$P_e \leq \sum_{X \in L, 0 < \|X\|_F \leq 2M} \frac{1}{(\det(I + \rho \theta^2 X X^*))^{n_r}}.$$

This discussion leads us to consider sums of the type

$$\sum_{X \in L, 0 < \|X\|_F \leq M} \frac{1}{(\det(I + c X X^*))^m}, \quad (2)$$

where  $c$  is considered a variable.

*Remark 1.1:* We remark that when  $c$  is very large, the terms in (2) are well-approximated by  $1/\det(c X X^*)^m$ . In the case  $T = n$ , we can consider sums of the type

$$S_L^m(M) := \sum_{X \in L(M)} \frac{1}{|\det(X)|^m}. \quad (3)$$

The asymptotic behavior of these sums, and its relation to the diversity-multiplexing trade-off of space-time codes, were analyzed in [2].

In this paper, we will address some additional aspects of MIMO space-time code optimization that are not captured by the approximate sums (3), but instead require to study the original sums (2). In particular, we will consider the following problems:

- Find upperbounds of the type

$$\sum_{X \in L, 0 < \|X\|_F \leq M} \frac{1}{(\det(I + c X X^*))^m} \leq c^{-k} f(M)$$

for some function  $f$  and positive constant  $k$ .

- How large should  $m$  be for the sum (2) to converge?

- What is the highest power  $k$  such that

$$\sum_{X \in L, \|X\|_F \leq M} \frac{1}{(\det(I + c X X^*))^m} \leq c^{-k} G$$

for some constant  $G$  and for every  $M$ ?

In the following we will give some general answers to these questions and build a framework for using these sums to analyze codes.

## II. DYADIC SUMMING AND UPPER BOUNDS FOR SHIFTED INVERSE DETERMINANT SUMS

Let's start by considering the decomposition of the shifted determinant. Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of  $XX^*$ . Then

$$\begin{aligned} \det(I + c X X^*) &= (1 + \lambda_1 c)(1 + \lambda_2 c) \cdots (1 + \lambda_n c) = \\ &= 1 + \binom{n}{1} p_1 c + \binom{n}{2} p_2 c^2 + \cdots + p_n c^n, \end{aligned}$$

where  $\binom{n}{i} p_i$  is the  $i$ -th symmetric polynomial of variables  $\lambda_1, \dots, \lambda_n$ . One should note that

$$p_1 = \text{Tr}(X X^*) = \|X\|_F^2 \quad \text{and} \quad p_n = \det(X X^*).$$

The following inequalities will be useful in the sequel:

*Proposition 2.1 (McLaurin's and Newton's inequalities):*

The coefficients  $p_i$  satisfy

$$\begin{aligned} p_1 &\geq \sqrt{p_2} \geq \sqrt[3]{p_3} \geq \cdots \geq \sqrt[n]{p_n}, \\ p_i^2 &\geq p_{i-1} p_{i+1}. \end{aligned} \quad (4)$$

*Corollary 2.2:* Let us suppose that  $\det(X X^*) \geq 1$ . With the previous notation we have that

$$p_k \geq \sqrt[2^{k-1}]{p_1},$$

for all  $n - 1 \geq k$ .

*Proof:* We have that

$$p_i \geq \sqrt{p_{i+1} p_{i-1}}.$$

Due to the condition  $p_n \geq 1$  we have that  $p_k \geq 1 \quad \forall k$ . Therefore

$$p_i \geq \sqrt{p_{i-1}}.$$

Induction now gives us the result.  $\square$

In the following we are interested in asymptotics and convergence and therefore we can forget the binomial terms and concentrate on the terms  $p_i$ . The following inequalities formalize this approach:

$$\begin{aligned} (\det(I + c X X^*))^m &= \\ &= \left( 1 + \binom{n}{1} p_1 c + \binom{n}{2} p_2 c^2 + \cdots + p_n c^n \right)^m \geq \\ &\geq (1 + p_1 c + p_2 c^2 + \cdots + p_n c^n)^m \geq \\ &\geq (c \|X\|_F^2 + c^n |\det(X X^*)|)^m = \\ &= \sum_{i=0}^m \binom{m}{i} c^{i+n(m-i)} \|X\|_F^{2i} |\det(X X^*)|^{m-i} \end{aligned}$$

In particular we have

$$\begin{aligned} & \sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m} \leq \\ & \leq \sum_{X \in L(M)} \frac{1}{c^{i+n(m-i)} \|X\|_F^{2i} |\det(XX^*)|^{m-i}} \end{aligned} \quad (5)$$

for every  $0 \leq i \leq m$ .

The following two Lemmas are useful to provide bounds for the sum in equation (5).

**Lemma 2.3 (Dyadic Summing):** Let  $f : \mathbb{R} \mapsto \mathbb{R}$  be a positive valued function, and  $I \subset \mathbb{R}$  be a discrete set. Suppose that there exist positive constants  $K$  and  $s$  such that  $\forall M \geq 1$ ,

$$\sum_{x \in I, 1 \leq x \leq M} f(x) \leq KM^s.$$

We then have that

$$\begin{aligned} \sum_{x \in I, 1 \leq x \leq M} \frac{f(x)}{x^t} &< K_1 & \text{if } t > s, \\ \sum_{x \in I, 1 \leq x \leq M} \frac{f(x)}{x^t} &< K_2 \log(M) & \text{if } t = s, \\ \sum_{x \in I, 1 \leq x \leq M} \frac{f(x)}{x^t} &< K_3 M^{s-t} & \text{if } t < s, \end{aligned}$$

for some constants  $K_1, K_2, K_3$  (depending on  $s$  and  $t$ ).

*Proof:* By partitioning the interval  $[1, M]$  into subintervals of the form  $[2^{i-1}, 2^i]$ , we get

$$\begin{aligned} \sum_{x \in I, 1 \leq x \leq M} \frac{f(x)}{x^t} &\leq \sum_{i=1}^{\lceil \log_2(M) \rceil} \sum_{x \in I, 2^{i-1} \leq x \leq 2^i} \frac{f(x)}{x^t} \leq \\ &\leq \sum_{i=1}^{\lceil \log_2(M) \rceil} \sum_{x \in I, 2^{i-1} \leq x \leq 2^i} \frac{f(x)}{2^{(i-1)t}} \leq \sum_{i=1}^{\lceil \log_2(M) \rceil} \frac{K 2^{is}}{2^{(i-1)t}} = \\ &= 2^t K \sum_{i=1}^{\lceil \log_2(M) \rceil} (2^{(s-t)i}). \quad \square \end{aligned}$$

**Lemma 2.4:** Let  $L \subset M_{n \times T}(\mathbb{C})$  be a lattice such that  $\|X\|_F \geq 1$  for all the non-zero points  $X \in L$ . Let  $g$  be a positive valued function defined in all the non-zero points of the lattice. If

$$\sum_{X \in L(M)} g(X) \leq KM^s$$

for some fixed positive constants  $K$  and  $s$ , then

$$\begin{aligned} \sum_{X \in L(M)} \frac{g(X)}{\|X\|_F^t} &< K_1 & \text{if } t > s, \\ \sum_{X \in L(M)} \frac{g(X)}{\|X\|_F^t} &< K_2 \log(M) & \text{if } t = s, \\ \sum_{X \in L(M)} \frac{g(X)}{\|X\|_F^t} &< K_3 M^{s-t} & \text{if } t < s, \end{aligned}$$

for some constants  $K_1, K_2, K_3$ .

*Proof:* This is simply the previous proposition applied to the function  $f(x) = \sum_{X \in L, \|X\|_F = x} g(X)$ .  $\square$

Note that the hypothesis  $\|X\|_F \geq 1$  in Lemma 2.4 doesn't incur any loss of generality since we can just rescale the lattice. However, in that case the constants  $K_1, K_2, K_3$  will depend on the scaling factor.

We can now obtain a set of upper bounds for shifted inverse determinant sums:

**Proposition 2.5:** Let us suppose that  $L$  is a  $k$ -dimensional lattice in  $M_{n \times T}(\mathbb{C})$ , such that  $\|X\|_F \geq 1$  for all the non-zero points  $X \in L$ , and that we have a bound

$$\sum_{X \in L(M)} \frac{1}{|\det(XX^*)|^t} \leq KM^{s(t)}.$$

We then have that

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m} \leq \min_{0 \leq i \leq m} \{W_i(M)\},$$

where for  $i \in \{0, \dots, m-1\}$  we have

$$\begin{aligned} W_i(M) &= \frac{G_i}{c^{i+n(m-i)}}, & \text{if } s(m-i) < 2i, \\ W_i(M) &= \frac{G_i}{c^{i+n(m-i)}} \log(M), & \text{if } s(m-i) = 2i, \\ W_i(M) &= \frac{G_i}{c^{i+n(m-i)}} M^{s(m-i)-2i} & \text{if } s(m-i) > 2i, \end{aligned}$$

where  $G_i$  are some constants. When  $i = m$ , we have  $W_m(M) = G_m c^{-m}$  if  $k < 2m$ ,  $W_m(M) = G_m c^{-m} \log M$  if  $k = 2m$  and  $W_m(M) = G_m c^{-m} M^{k-2m} M$  if  $k > 2m$ .

*Proof:* The conclusion follows from equation (5) and from Lemma 2.4 with  $g(X) = 1/\det(XX^*)^{m-i}$ . For the special case  $i = m$ , observe that the number of lattice points in  $L(M)$  is proportional to the volume of the ball of radius  $M$  in  $\mathbb{R}^k$ .  $\square$

As a consequence of Proposition 2.5 in the case  $i = m$ , the shifted determinant sum will converge for  $m > k/2$ :

**Proposition 2.6 (Convergence):** Let us suppose that  $L$  is a  $k$ -dimensional lattice in  $M_{n \times T}(\mathbb{C})$  such that  $\|X\|_F \geq 1$  for all the non-zero points  $X \in L$ . We then have that

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^{k/2+\epsilon}} \leq G_\epsilon c^{-k/2},$$

where  $\epsilon$  is any positive number and  $G_\epsilon$  a constant independent of  $M$ , but dependent on  $\epsilon$ .

We can conclude that while

$$\sum_{X \in L(M)} \frac{1}{\det(XX^*)^m}$$

does not usually converge for any  $m$  [2], quite the opposite is true for the sum

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m}.$$

As long as the power  $m$  is large enough this sum will always converge.

### III. SHIFTED INVERSE DETERMINANT SUMS AND SNR LEVEL ANALYSIS

Let us introduce one more use for shifted inverse determinant sums. If we have a finite space-time NVD lattice code in  $M_n(\mathbb{C})$ , then in the high SNR regime the diversity order is  $nn_r = n_t n_r$ . However this regime is rarely visible on error performance curves. We will now see how shifted inverse determinant sums can explain this behavior, and provide an estimate of the SNR threshold beyond which higher diversity kicks in.

*Proposition 3.1:* Suppose that we have a  $k$ -dimensional lattice  $L \in M_{n \times T}(\mathbb{C})$  such that a determinant sum upper bound

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m} \leq KM^t c^{-d},$$

holds for some constants  $d, K$  and  $t$ . Then the average error probability is upper bounded as

$$P_e(\rho) \leq KM^{d+t} \rho^{-d}, \quad (6)$$

when transmitting with signal-to-noise ratio  $\rho$ .

*Proof:* The average energy of the code  $\frac{1}{\sqrt{M}}L(M)$  is less than 1. Let us now suppose that  $\theta > 1$  is such a constant that  $\theta \frac{1}{\sqrt{M}}L(M)$  has average energy 1. We then have that

$$\begin{aligned} P_e(\rho) &\leq \sum_{X \in L(M)} \frac{1}{(\det(I + \rho\theta^2 XX^*))^m} \\ &\leq KM^{t+d}(\rho\theta^2)^{-d} = KM^{t+d} \rho^{-d}, \end{aligned} \quad (7)$$

which concludes the proof.  $\square$

This result has several implications. The first is that we can estimate the SNR threshold beyond which we can see diversity order  $d$ . We can see from equation (7) that when  $SNR = \rho \geq K' M^{(t+d)/d}$  the diversity  $d$  will appear; before this point we don't have guaranteed diversity  $d$ .

Another implication is easier to explain through an example.

*Example 3.1:* Let us suppose that we have an 8-dimensional lattice code  $L$  in  $M_2(\mathbb{C})$  and bounds

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^4} \leq \min\{K_1 M^4 c^{-8}, K_2 c^{-4}\}.$$

In order to have guaranteed diversity 8 we must have  $\rho \geq K'_1 M^{3/2}$ . For guaranteed diversity 4 the SNR condition is  $\rho \geq K'_2 M^1$ . This shows that when the code grows we need eventually considerably more energy to have guaranteed diversity 8, independently of the size of the constants  $K_1$  and  $K_2$ .

### IV. SHIFTED INVERSE DETERMINANT SUMS AND DMT ANALYSIS

It was proven in [2] that the growth of the inverse determinant sums of a lattice code  $L \in M_n(\mathbb{C})$  describes the diversity-multiplexing gain trade-off (DMT) [3] of the code  $L$  for multiplexing gains  $r \in [0, 1]$ . We will now show that

the shifted determinant sum bounds are useful to analyze the DMT of a code for higher multiplexing gains. We will also show how we can use these sums to evaluate the DMT of a lattice code under naive lattice decoding.

#### A. Lower bounds for the DMT under ML decoding

*Definition 4.1:* Given the lattice  $L \subset M_{n \times T}(\mathbb{C})$ , a space-time lattice coding scheme associated with  $L$  is a collection of STBCs where each member is given by

$$C_L(\rho) = \rho^{-\frac{rT}{k}} L \left( \rho^{\frac{rT}{k}} \right) \quad (8)$$

for the desired multiplexing gain  $r$  and for each  $\rho$  level.

*Proposition 4.1:* Let  $L$  be a  $k$ -dimensional lattice in  $M_{n \times T}(\mathbb{C})$  and suppose that the determinant sum upper bound

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m} \leq Kc^{-a} f(M),$$

holds for some positive constants  $K$  and  $a$ . We then have that for SNR  $\rho$  the average error probability of the code  $C_L(\rho)$  has an upperbound

$$P_e(\rho) \leq K_1 \rho^{-a+2arT/k} f(2\rho^{\frac{rT}{k}}).$$

*Proof:* The average energy for the code  $C_L(\rho)$  is less than 1. Now for transmission with SNR  $\rho$ , each of the codewords gets multiplied with  $\rho^{1/2}$  as in (1). We now have

$$P_e(\rho) \leq \sum_{X \in L(2\rho^{\frac{rT}{k}})} \frac{1}{(\det(I + cXX^*))^m},$$

where  $c = \rho^{1-2rT/k}$ . The final result is then simply gotten by substitution.  $\square$

The following DMT bound is a direct corollary of the previous result:

*Corollary 4.2:* Let us suppose that we have an upperbound

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m} \leq Kc^{-a} M^b.$$

We then have that the DMT of the code  $L$  is lowerbounded by the following line:

$$[r, (a - rT(2a + b)/k)^+].$$

We also have the following curiosity, which shows that any full dimensional lattice achieves the full multiplexing gain, when we have enough receive antennas:

*Corollary 4.3:* Let  $L$  be a  $2nT$ -dimensional lattice code in  $M_{n \times T}(\mathbb{C})$ . If  $n_r > nT + 1$ , the code  $C_L(\rho)$  has an upperbound for the average error probability

$$P_e \leq \rho^{(1-r/n)n_r},$$

when transmitting with SNR  $\rho$ .

*Proof:* This follows from Proposition 2.5 with  $m = n_r > k/2$ , yielding  $a = n_r$  and  $b = 0$  in Corollary 4.2.  $\square$



### B. Lower bound for the DMT under naive lattice decoding

Let us consider naive lattice decoding as defined in [7], which consists in minimizing the Euclidean metric with respect to the received signal over all the lattice points  $X' \in L$ , regardless of whether they belong to the finite code.

It is clear that the average probability of error of naive lattice decoding can be upper bounded by a determinant sum over the whole lattice. This bound is relevant only when the sum is converging. We thus state the following Proposition, which can be proven in the same way as Corollary 4.2:

*Proposition 4.4:* Let  $L \in M_{n \times T}(\mathbb{C})$  be a  $k$ -dimensional lattice, and suppose that a determinant sum upper bound

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m} \leq Kc^{-a}, \quad (9)$$

holds for some positive constants  $K$  and  $a$ . Then the DMT of the code  $C_L(\rho)$ , under naive lattice decoding, is lower bounded by

$$[r, (a - 2rTa/k)^+].$$

### V. EXAMPLES

#### A. Analyzing the Golden code

Let us consider the Golden code  $L \subset M_2(\mathbb{C})$ , which is an 8-dimensional lattice. According to [8], if  $n_r > 1$  we have

$$\sum_{X \in L(M)} \frac{1}{|\det(X)|^{2n_r}} \leq KM^4, \quad (10)$$

where  $K$  is a positive constant.

With the previous notation we have that

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^4} \leq \min_{i \in \{0,2,4\}} W_i(M),$$

where  $W_0 = K_0c^{-8}M^4$ ,  $W_2 = K_2c^{-6}\log(M)$  and  $W_4 = K_4c^{-4}$  for some constants  $K_i$ .

*Proposition 5.1:* Under naive lattice decoding, when received with 4 antennas, the Golden code achieves the DMT curve

$$[r, (2(2 - r))^+].$$

With ML the Golden code achieves the DMT curve

$$[r, \max\{(8 - 5r), (6 - 3r)\}^+],$$

which coincides with the optimal DMT  $[r, (4 - r)(2 - r)^+]$ .

We can see that even with naive lattice decoding the Golden code does achieve the optimal multiplexing gain. However, the maximal diversity is only 4.

*Remark 5.1:* Here one should note that for multiplexing gains  $r \in [0, 1]$  the sum

$$\sum_{X \in L(M)} \frac{1}{(\det(cXX^*))^4},$$

does provide the best upper bound. However, when  $r \in [1, 2]$  the sum

$$\sum_{X \in L(M)} \frac{1}{(c^6 \|X\|_F^4 \det(XX^*))^2},$$

gives a tighter upper bound.

### B. Analyzing diagonal number field codes

Let us now consider a complex diagonal number field code. Such a code is  $2n$ -dimensional NVD lattice in  $M_n(\mathbb{C})$ . As proved in [2] we have that for  $m \geq 1$  we have

$$\sum_{X \in L(M)} \frac{1}{|(\det(XX^*))|^m} \leq K \log(M)^{3n-1}, \quad (11)$$

for some constant  $K$ .

*Proposition 5.2:* Let  $L$  be a diagonal number field code in  $M_n(\mathbb{C})$  such that  $\det(XX^*) \geq 1$  for all  $X \in L$ ,  $X \neq \mathbf{0}$  and let  $m > 1$ . We then have that

$$\sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m} \leq Kc^{-nm+1},$$

where  $K$  is some constant independent of  $M$ .

*Proof:* We begin with

$$\begin{aligned} \sum_{X \in L(M)} \frac{1}{(\det(I + cXX^*))^m} &\leq \sum_{X \in L(M)} \frac{1}{(c^n p_n + c^{n-1} p_{n-1})^m} \\ &\leq c^{-mn+1} \sum_{X \in L(M)} \frac{1}{(p_n^{m-1} p_{n-1})} \\ &\leq c^{-mn+1} \sum_{X \in L(M)} \frac{1}{(p_n^{m-1} (p_1^{1/2(n-2)}))}, \end{aligned}$$

where the last equation follows from Corollary 2.2. As  $p_1 = \|X\|_F^2$  we can then apply Lemma 2.4.  $\square$

As a Corollary to the previous we have the following:

*Proposition 5.3:* Under naive lattice decoding the number field code achieves the DMT curve

$$[r, ((n_t n_r - 1)(1 - r))^+].$$

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